MINIMAL SURFACES IN HYPERBOLIC GEOMETRY

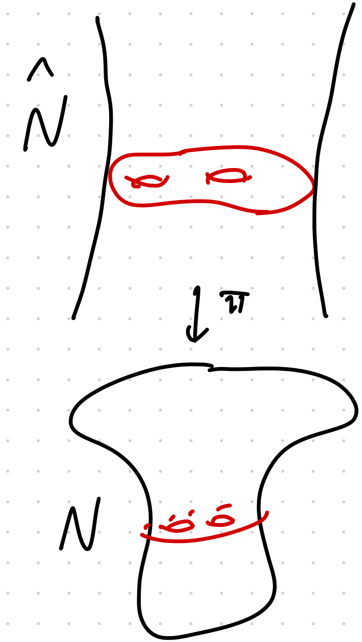
Winter School Côte d'Azur

Lecture I, 6th January 2026

Hyperbolic manifolds

Let (N, h) closed hyperbolic manifold, with $\Gamma \cong \pi_1 S$ a surface subgroup.

Let $\hat{N} \xrightarrow{\pi} N$ the covering such that $\pi_*(\pi_1 \hat{M}) = \Gamma$.



Fact If M is a 3-manifold such that $\pi_1 M \cong \pi_1 S$, then $M \cong S \times \mathbb{R}$.

Consider

$$AH(S) = \left\{ \begin{array}{c} \text{complete hyperbolic} \\ \text{metrics on } S \times \mathbb{R} \end{array} \right\} \Big/ \mathrm{Diffeo}_0(S \times \mathbb{R}) \ni [(\hat{N}, \pi^* h)]$$

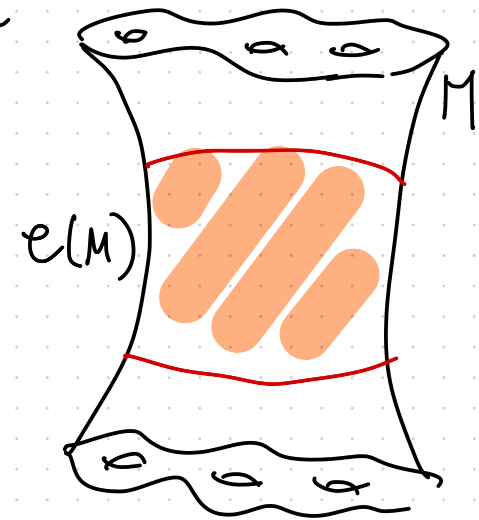
have topologies; geometric/algebraic convergence

The interior of $AH(S)$ is a ball of $\dim_{\mathbb{R}} = 12g-12$

Its elements are quasi-Fuchsian manifolds

<u>Def</u> Let $(M \overset{S\times\mathbb{R}}{\cong}, h)$ be a complete hyperbolic manifold.

M is quasi-Fuchsian if it contains a geodesically convex, compact subset



$\mathcal{C}(M)$ = convex core of M

= smallest such subset

Schoen–Yau, Sacks–Uhlenbeck:

existence of $\Sigma \simeq S \times \{*\}$ minimal

(Actually, all contained in $\mathcal{C}(M)$).

not unique in general!

Uhlenbeck's seminal paper (1983)

Recall that

- $B = -\nabla N$ is the shape operator of $\Sigma \subset M$
- the eigenvalues λ, μ of B are the principal curvatures
- Σ is minimal if $\lambda + \mu = 0$

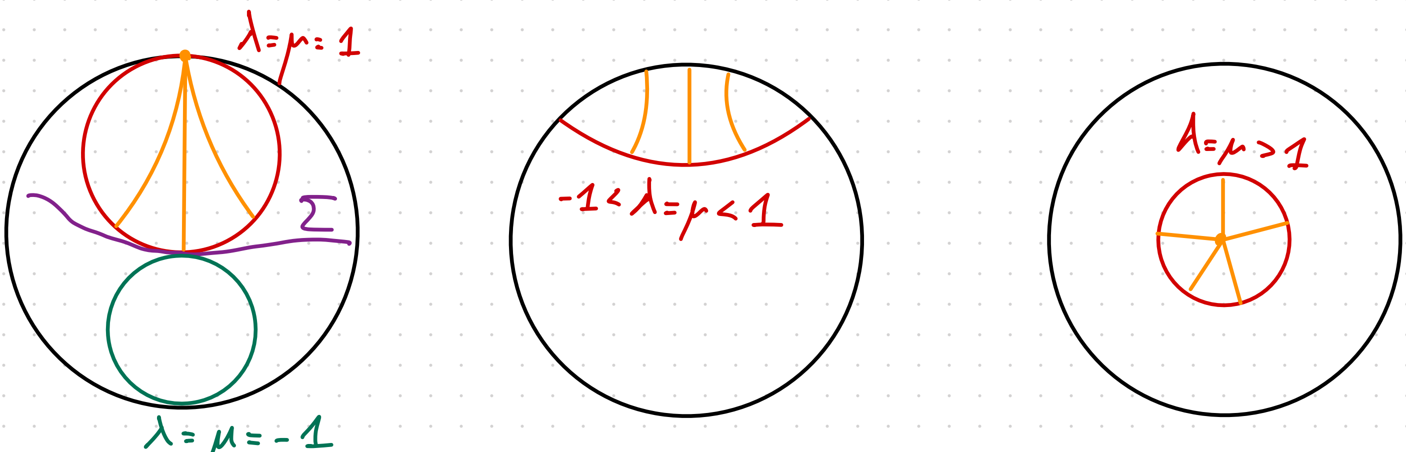
Def Let $(M \cong S \times \mathbb{R}, h)$ be a complete hyperbolic manifold. M is (weakly) almost-Fuchsian if it contains a closed minimal surface such that the principal curvatures are in $(-1,1)$ $([-1,1])$

Thm (Uhlenbeck)

If M is weakly almost-Fuchsian, then it contains a unique minimal surface homotopic to $S \times \{*\}$.

The condition $|\lambda|, |\mu| \leq 1$:

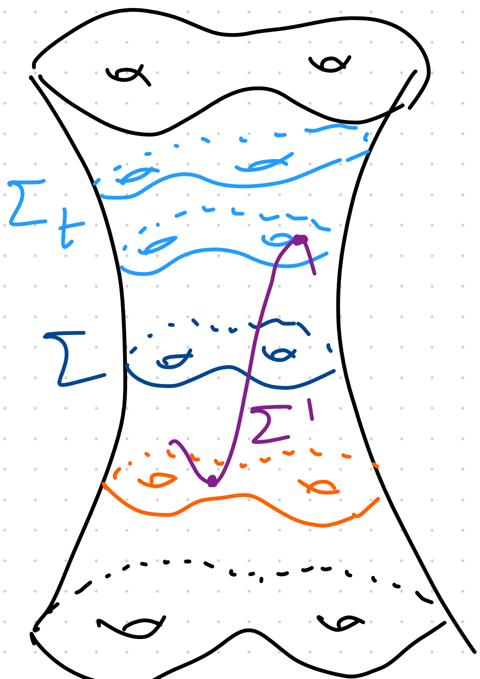
"Σ is less curved than a horosphere"



Sketch of Pf:

The map $\Sigma \times \mathbb{R} \longrightarrow M$ is a diffeo

$(p, t) \longmapsto \exp_p(tN(p))$



$t > 0$, $H_{\Sigma_t} < 0$

$t < 0$, $H_{\Sigma_t} > 0$

Geometric maximum principle

$\Downarrow$

uniqueness of Σ

See Sheet 2, Exercises 2 & 3

Summary: WAF $\Longrightarrow$ uniqueness of minimal surface

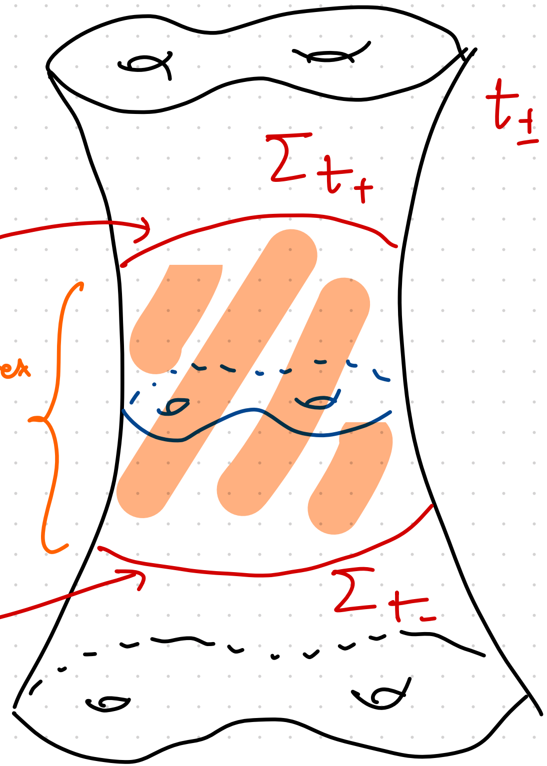
not a necessary condition!

Thm (Huang-Lowe '25)
There exist quasi-Fuchsian manifolds that contain a unique closed minimal surfaces homotopic to $\Sigma \times \{*\}$, that are not weakly almost-Fuchsian.

Thm (Uhlenbeck)

If M is almost-Fuchsian, then it is quasi-Fuchsian.

Sketch of Pf:



$$t_\pm = t_\pm \left(1 - \sup_{p\in\Sigma}\{|\lambda(p)|, |\mu(p)|\}\right)$$

concave

geodesically convex & compact

convex

Rmk: only needs $|\lambda|, |\mu| < 1$, not minimal.

Q! What about WAF ?

Theorem (Nguyen - Schlenker - S. '25)

Let (M, h) be a hyperbolic manifold, let $S \subset M$ be an embedded, orientable, two-sided closed minimal surface with principal curvatures in $[-1, 1]$.

Then any neighbourhood U of S in M contains a (non-minimal) surface with principal curvatures in $(-1, 1)$.

Cor If M is weakly almost-Fuchsian, then it is quasi-Fuchsian.

Def Let $(M \overset{\sim}{=} S \times \mathbb{R}, h)$ be complete hyperbolic.
M is **nearly-Fuchsian** if it contains a closed surface with principal curvatures in $(-1,1)$

from 2000s

Cor The conjecture "AF $\Longleftrightarrow$ NF" is false.

$$\begin{tikzcd}[column sep=large, row sep=large]
 & & NF \arrow[dr, Rightarrow, shift left] \arrow[dl, Rightarrow, shift left, "\times"', "\text{openness}"] & \\
AF \arrow[r, Rightarrow, shift right] & WAF \arrow[l, Rightarrow, shift right, "\times"', "\text{Uhlenbeck}"'] \arrow[ur, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] & & QF \arrow[ul, Rightarrow, shift left, "\times", "\text{Davalo '24}"'] \\
 & & \text{uniqueness} \arrow[ul, Rightarrow, shift left, "\times", "\text{HL}"] &
\end{tikzcd}$$